LECTURE 13

Front form QED(3+1): The spin-multiplet structure of the positronium spectrum at strong coupling*

U. Trittmann

Max–Planck-Institut für Kernphysik
Postfach 10 39 80, D–69029 Heidelberg, Germany

1. INTRODUCTION

The practitioner has to think about (at least) two obstacles, when he wants to solve a Hamiltonian field theory problem in *front form* dynamics. One is the problem that the theory is not manifestly rotational invariant. The other is the question, how one should construct effective theories at all in this framework. Of course, the first problem exists analogously in instant form dynamics, where the theory is not manifestly Lorentz invariant. Connected to this problem is the fact that any symmetry which is not manifest will break down immediately with the slightest approximation in a Hamiltonian field theory. There is broad agreement that the construction of effective theories is inevitable, if one wants to describe the low energy region, *i.e.* the bound states, of a strongly coupled theory. A variety of formalisms were suggested concerning this topic [1][2][3].

In this article I shall present quantitative investigations concerning the above two problems with a specified model, namely positronium as a QED(3+1) bound state. The model is represented by the energy diagrams of Fig. 1. To be more specific, an effective theory was constructed following the work of Pauli[3]. The effective matrix elements were calculated analytically, and the

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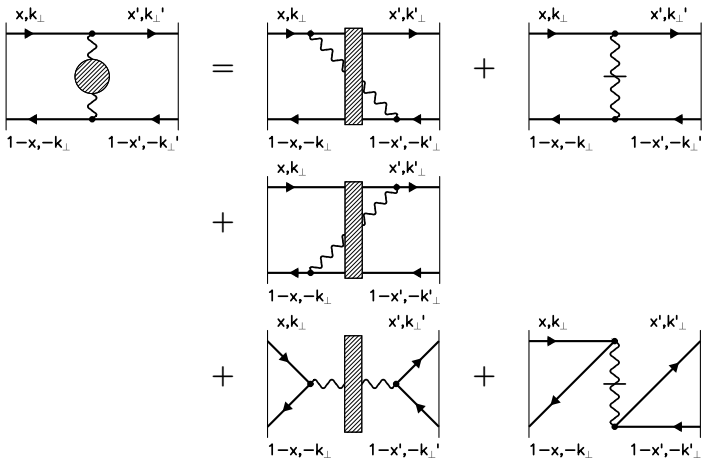

Fig. 1. — The graphs of the positronium model. Effective photon lines are labeled by hashed rectangles. The graphs of the annihilation interaction are at the bottom line.

emerging integral equation was put on the computer to numerically solve for the eigenvalues and eigenfunctions of the Hamiltonian.

Proceeding in this manner, one can answer the following questions *quantitatively*: Are the results of front form theory rotationally invariant, *i.e.* do states of different J_z form multiplets? How good is the underlying effective theory concerning its cutoff dependence? Is the multiplet structure of the spectrum reproduced correctly? And: what is the numerical evidence for the degeneracy of corresponding states?

2. METHOD

A complete description of the applied method is surely beyond the scope of this article. I review shortly its major steps.

To get a finite dimensional Hamiltonian out of the infinite dimensional canonical Hamiltonian, we introduce a cutoff Λ on the kinetic energies. We then (formally) map an $N \times N$ onto a 2×2 block matrix by subsequent projections (Bloch-Feshbach formalism) and call these sectors *P-space* and *Q-space*. Note that the Q-space is already an effective sector by construction:

$$H_Q = H_Q^{(N)} + X_Q^{\text{eff}},$$

where $H_Q^{(N)}$ is a piece of the original Hamiltonian and X_Q^{eff} are the interactions

generated by the projections. The Hamiltonian reads

$$H_{\rm LC} = \begin{pmatrix} H_P & H_{PQ} \\ H_{QP} & H_Q \end{pmatrix}.$$

The final projection maps the Q-space onto the P-space. One solves for the state $\hat{Q}|\psi\rangle = |e\bar{e}\gamma\rangle$ with a resolvent involving the *redundant parameter* ω

$$G(\omega) := \langle Q|\omega - H_{\rm LC}|Q\rangle^{-1},$$

and obtains the *nonlinear* equation

$$H^{\rm eff}_{\rm LC}(\omega)|\psi_n(\omega)\rangle = M_n^2(\omega)|\psi_n(\omega)\rangle,$$

with the effective Hamiltonian ($\hat{P}|\psi\rangle = |e\bar{e}\rangle$)

$$H^{\rm eff}_{\rm LC}(\omega) := \hat{P}H_{\rm LC}\hat{P} + \hat{P}H_{\rm LC}\hat{Q}(\omega - H_{\rm LC})^{-1}\hat{Q}H_{\rm LC}\hat{P}.$$

The fixing of the redundant parameter works as follows. We divide the sector Hamiltonian in Q-space into $H_Q = M_Q^2 + V_Q$, and the interaction into a diagonal part $\langle V_Q\rangle$ and a non-diagonal part δV_Q

$$V_Q = \langle V_Q\rangle + \delta V_Q.$$

With the definition $T^* := \omega - \langle V_Q\rangle$ we expand the resolvent around the diagonal interaction $\langle V_Q\rangle$

$$\begin{aligned} \frac{1}{\omega - H_Q} &= \frac{1}{T^* - M_Q^2} + \frac{1}{T^* - M_Q^2}\delta V_Q \frac{1}{T^* - M_Q^2 - \delta V_Q} \\ &\simeq \frac{1}{T^* - M_Q^2}. \end{aligned} \tag{1}$$

The approximation in the last step is to consider the first term only. It has severe consequences: a collinear singularity occurs, proportional to

$$\frac{1}{\mathcal{D}}\frac{\Delta(x, k_\perp, x', k'_\perp; T^*)}{|x - x'|}, \qquad with \quad \mathcal{D}(x, x'; T^*) := |x - x'|(T^* - M_Q^2).$$

We can calculate $\Delta(x, k_\perp, x', k'_\perp; T^*)$ with P-space graphs

$$\Delta = M_Q^2 - \omega - \frac{(k_\perp - k'_\perp)^2}{x - x'} - \frac{1}{2}\left(l_e^- - l_{\bar{e}}^-\right),$$

where $l_e^\mu := (k'_e - k_e)^\mu$ and $l_{\bar{e}}^\mu := (k_{\bar{e}} - k'_{\bar{e}})^\mu$ are the momentum transfers. One determines the parameter ω by demanding that the *collinear* singularity, induced by the approximation, Eq. (1), vanishes

$$\Delta(x, k_\perp, x', k'_\perp; T^*) = 0, \qquad \forall x, x', k_\perp, k'_\perp.$$

This allows for the calculation of an explicit expression for T^*

$$T^*(x,\vec{k}_\perp;x',\vec{k}'_\perp) = \frac{1}{2}\left(\frac{m_f^2+\vec{k}_\perp}{x(1-x)} + \frac{m_f^2+\vec{k}'_\perp}{x'(1-x')}\right).$$

The interpretation is that T^* is an approximation of the summed interactions of the higher Fock states.

Finally, the procedure yields an integral equation. The effective Hamiltonian operates only in P-space, and the continuum version of the eigenvalue problem is

$$\begin{aligned} 0 \;=\; & \left(\frac{m_f^2+\vec{k}_\perp^2}{x(1-x)} - M_n^2\right)\psi_n(x,\vec{k}_\perp;\lambda_1,\lambda_2) \\ & \frac{g^2}{16\pi^3}\sum_{\lambda'_1,\lambda'_2}\int_D \frac{dx'd^2\vec{k}'_\perp}{\sqrt{xx'(1-x)(1-x')}}\frac{j^\mu(l_e,\lambda_e)j_\mu(l_{\bar{e}},\lambda_{\bar{e}})}{l_{\bar{e}}^\mu l_{e,\mu}} \\ & \times\psi_n(x',\vec{k}'_\perp;\lambda'_1,\lambda'_2), \end{aligned}$$

with an integration domain D defined by a cutoff on the kinetic energy of the states. One observes that the effective interaction is gauge invariant and a Lorentz scalar

$$U_{\text{eff}} := \frac{j^\mu(l_e,\lambda_e)j_\mu(l_{\bar{e}},\lambda_{\bar{e}})}{l_{\bar{e}}^\mu l_{e,\mu}}.$$

3. RESULTS

As pointed out in the last section, the goal is to construct an effective Hamiltonian describing the positronium bound states. We showed how to find a reasonable *ansatz* for contributions of higher Fock states. To find out about the rotational symmetry of the theory, one has to observe that the rotation operator $\mathcal{J}_3$ around the z axis is kinematic. Consequently, states can be classified according to the corresponding quantum number J_z. We have shown in Refs. [4][5] that closed expressions for the matrix elements of the *effective Hamiltonian* can be calculated, even for arbitrary J_z. This enables one to solve for the positronium eigenvalues in each sector of J_z separately and to compare the results to find out about possible degeneracies. A computer code for solving these eigenvalue problems, *i.e. integral equations*, has been generated with correct *Coulomb counterterms*. The latter is important to guarantee numerical stability and precision high enough to be able to give a quantitative statement concerning the (non)degeneracy of states. In fact, the convergence of the eigenvalues turns out to be exponential in the number of integration points. It is plotted in Fig. 4.

The results are compared to equal time perturbation theory (ETPT). The spectrum is shown in Fig. 2, which is the most important of this article. A one-to-one comparison between the multiplets for a Bohr quantum number $n = 2$

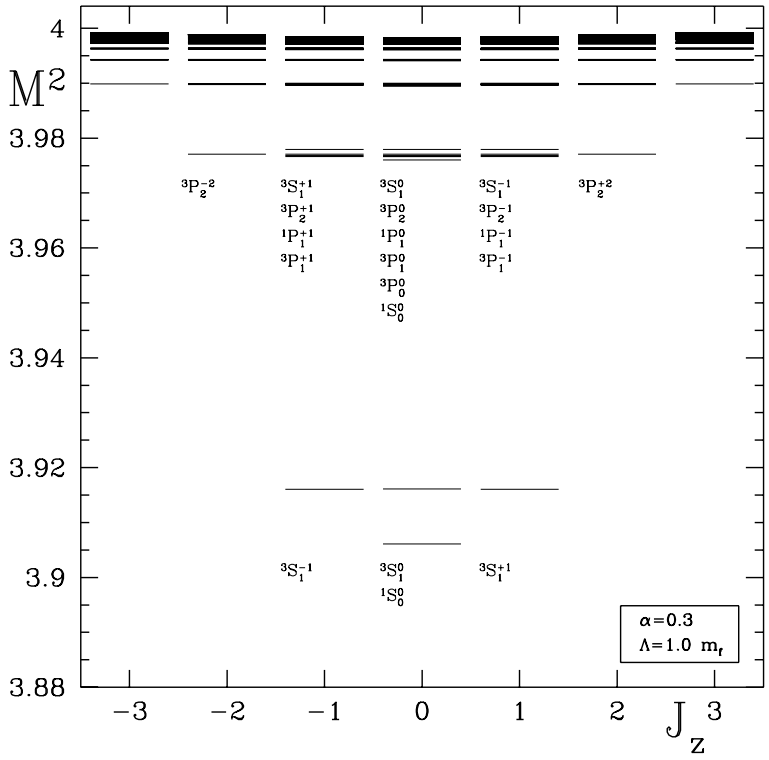

Fig. 2. — Positronium spectrum in different J_z sectors. $N_1 = N_2 = 21$.

is depicted in Fig. 3. It is quite clear how to interpret these results. One reads off easily from the number of degenerate J_z levels the quantum number J of the complicated operator J^2, consisting of the angular momentum operators, *cf.* the theory of the Poincaré group in front form dynamics [4, Chapter 3.1].

It is worthwhile mentioning another non-trivial result of the formalism considered in this article. The annihilation channel can be included straightforwardly, which is an intrinsic check of the effective theory. Surprisingly the instantaneous and the dynamic graphs act in different J_z sectors. An interesting feature concerning the annihilation channel is the calculation of the hyperfine splitting, where this channel plays a major role and yields a well-known contribution. The coefficient of the hyperfine splitting is defined as

$$C_{hf} \quad = \quad \frac{M_t - M_s}{\alpha^4} = \frac{1}{2}\left[\frac{2}{3} + \left(\frac{1}{2}\right) - \frac{\alpha}{\pi}\left(\ln 2 + \frac{16}{9}\right) + \mathcal{O}(\alpha^2)\right].$$

The results for the two formalisms are

$$C_{hf} = \left\{ \begin{array}{rl} 0.56 & ;\text{this work} \\ \frac{7}{12} \simeq 0.58 & ;\text{ETPT}[\mathcal{O}(\alpha^4)], \end{array} \right.$$

which are in reasonable agreement.

Let us have a closer look at the results. One question listed above was, how good is the numerical evidence for degeneracies. One can plot the deviation of corresponding eigenvalues for J_z=0 and J_z=1 multiplets with growing number of integration points, as was done in Fig. 12 of Ref. [5]. A χ^2 fit of the difference function to

$$\Delta M^2(N) = a - b \exp\left\{(N - N_0)/c\right\}.$$

yields for the ground state triplet $1\,^3S_1$: $a = -(5.47 \pm 0.95) \times 10^{-5}$, $b = (1.88 \pm 0.03) \times 10^{-4}$, $c = 4.03 \pm 0.11$. It seems thus quite justified to consider these states degenerate.

The cutoff dependence of the eigenvalues was investigated, too. For the actual graphs, see Ref. [6]. The role of the annihilation channel seems to be to stabilize the eigenvalues. For the triplet ground state we can fit the dependence on *log* Λ with

$$M_t^2(\Lambda) = \begin{cases} 3.90976 - 0.01858 \log\Lambda + 0.00789 \log^2\Lambda & ;\text{no annih.} \\ 3.91392 - 0.00029 \log\Lambda + 0.00015 \log^2\Lambda & ;\text{incl. annih.} \end{cases}$$

The decrease of the triplet with $\log\Lambda$ is suppressed by the inclusion of the annihilation channel by a factor of 60! Care is, however, to be taken, since the annihilation channel is absent in all QED bound states not built out of a particle and its antiparticle [7].

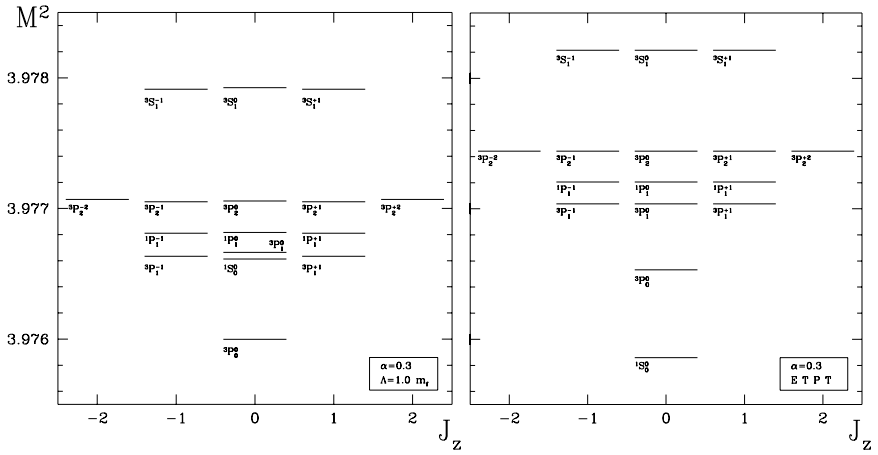

Fig. 3. — Comparison of multiplets for n=2: (a) results of the present work with $\alpha = 0.3$, $\Lambda = 1.0\, m_f$, $N_1 = N_2 = 21$; (b) equal-time perturbation theory (ETPT) up to order $\mathcal{O}(\alpha^4)$.

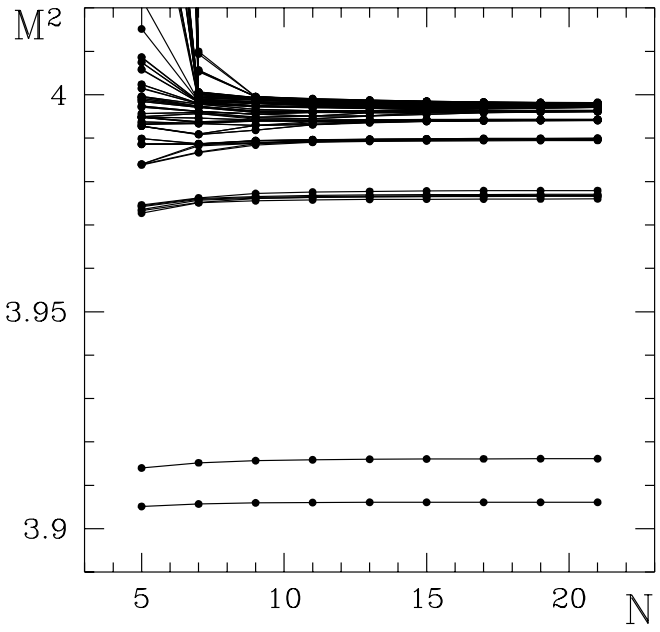

Fig. 4. — The positronium spectrum including the annihilation channel in the J_z=0 sector. Parameters of the calculation: $\alpha = 0.3, \Lambda = 1.0\, m_f$. The mass squared eigenvalues M_n^2 in units of the electron mass m_f^2 are shown as functions of the number of integration points $N \equiv N_1 = N_2$. The triplet states, especially 1^3S_1, are lifted up, the singlet mass eigenvalues are the same as without the annihilation channel.

4. CONCLUSIONS

To conclude, we review the results of the positronium theory in *front form* dynamics as presented in this article.

The correct positronium spectrum is obtained in all J_z sectors. The J_z states form (degenerate) multiplets, which means that rotational invariance is restored in the solution of a Hamiltonian field theory in *front form* dynamics. Consequently, there is no need to diagonalize the complicated rotation operator $\mathcal{J}^2$, because its eigenvalues can be read off by counting the number of degenerate eigenstates within the J_z multiplets. The annihilation channel can be introduced into the theory straightforwardly. Furthermore, the annihilation channel seems important for the cutoff behavior. From these facts, we can deduce that the applied underlying effective theory is correct. The computer code which was generated to solve for the eigenstates is applicable also to other effective Hamiltonians. The application to WEGNER'S formalism [2] is work in progress. As an outlook one could think of plugging a running coupling constant into the code and to calculate the meson spectrum. Surely, this implies a deeper

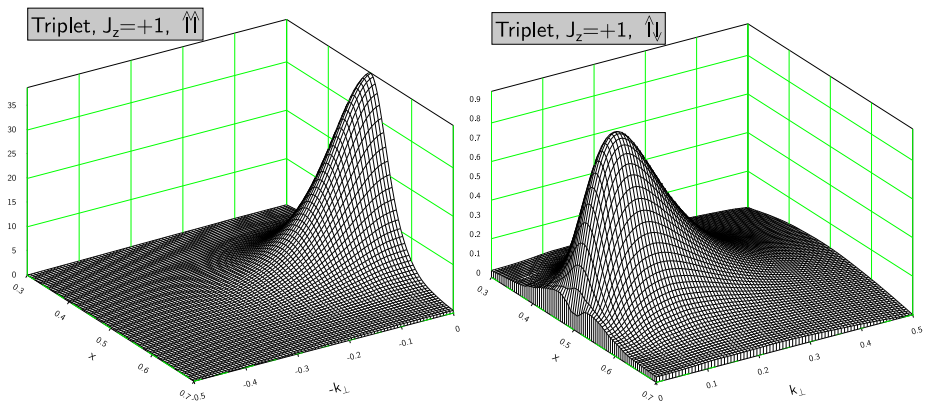

Fig. 5. — The triplet ground state wavefunction for $J_z = +1$ as a function of the longitudinal momentum fraction x and the transverse momentum $k_\perp$, omitting the dependence on the angle φ. The calculation was done with $\alpha = 0.3, \Lambda = 1.0\, m_f, N_1 = 41, N_2 = 11$. Shown are: (a) $(\uparrow\uparrow)$-component, (b) $(\uparrow\downarrow)$-component.

understanding of the numerical behavior of the occuring severe singularities.

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